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AN ALGORITHMIC SOLUTION FOR A QUEUEING MODEL OF A COMPUTER SYSTEM WITH INTERACTIVE AND BATCH JOBS

Syracuse University

J. G. Shanthikumar
Robert G. Sargent

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APPROVED: Clement D. Falzarano

CLEMENT D. FALZARANO
Project Engineer

APPROVED:

JOHN J. MARCINIAK, Colonel, USAF
Chief, Information Sciences Division

FOR THE COMMANDER: John P. Huss

JOHN P. HUSS
Acting Chief, Plans Office

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The "Language Studies" contract is divided into four project areas, all of which are directed to the problems of effectively, reliably and efficiently using modern computers in a wide range of applications.

Three of the projects deal with methods of communicating with computers.

Task 1. Very High Level Programming Systems (P.I: J.A. Robinson). This group is working towards combining the features developed to support work in the area of artificial intelligence and those used in general → next page

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program development into a new conceptual framework that can be understood and used by a large community of users.

Task 2. Proving Program Correctness (P.I: J.C. Reynolds). This group is working towards programming language designs which increase the probability that specification errors will be detected by the compiler or interpreter and to provide the language facilities so that users will more nearly be able to prove that programs perform as they are specified than is currently possible.

Task 3. Grammars of Programming (P.I: E.F. Storm). This group is working towards the development of methods which will allow users to communicate with computer programs in terms more normal to their every day communication forms.

Task 4. Systems Studies (P.I: R.G. Sargent). This group is working towards developing more sophisticated and efficient models of computer systems which can predict system performance when given particular parameter values. The current efforts concern models of transaction processing systems (TPS).

Preface

This report describes efforts completed in the Language Studies project at Syracuse University under RADC contract F30602-77-C-0235. The work covers the period October 1, 1977 through September 30, 1980.

The report is produced in five volumes to facilitate single volume distribution.

Volume 1. Report from the Very High Level Programming Systems task. Report title is "Logic Programming in Lisp".

Volume 2. Report from the Systems Studies task. Report title is "Multiple Finite Queueing Model with Fixed Priority Scheduling".

Volume 3. Report from the Systems Studies task. Report title is "An Algorithmic Solution for a Queueing Model of a Computer System with Interactive and Batch Jobs.

Volume 4. Report from the Grammars of Programming task. Report title is "Programming Control Structures in a High Level Language.

Volume 5. Report from the Proving Program Correctness task. Report title is "Realignment".

AN ALGORITHMIC SOLUTION FOR A QUEUEING MODEL OF A COMPUTER SYSTEM WITH INTERACTIVE AND BATCH JOBS

ABSTRACT

A queueing model with two customer classes, one with infinite and the other with finite source, is used as a model for a computer system with interactive and batch jobs. Using an imbedded Markov Chain representation of this queueing system, and an algorithmic approach, the steady state joint probability distribution of the number of interactive and batch jobs at a job service completion epoch is derived. Server utilization, mean waiting times, joint probability distribution, and mean number of interactive and batch jobs at an arbitrary time epoch are derived using these probabilities, discrete state level crossing analysis and Little's result.

Key Words: Computer Systems Modelling, Priority Queues, Imbedded Markov Chain, Algorithmic Solution, Level Crossing Analysis.

1. INTRODUCTION

The intent of this paper is to present an algorithmic solution for a queueing model of a computer system with interactive and batch jobs. Interactive jobs arrive from a finite number of interactive terminals and batch jobs arrive from an infinite size population. Once an interactive job arrives to the computer system, the corresponding terminal stays passive until the job is processed, at which time the terminal becomes active and begins the process of submitting a new job.

The queueing models associated with this system are special cases of priority queues with two classes of customers with one finite and the other infinite sources. Models in which the finite source jobs having preemptive priority over infinite source jobs have been extensively analyzed by Avi-Itzhak and Naor [1], Colard and Latouche [2], Jaiswal [4], Jaiswal and Thiruvengadam [5], and Thiruvengadam [12]. While the technique used in [1, 4, 5, 12] is complex and obtaining numerical results is difficult, the algorithmic approach discussed by Colard and Latouche [2] is very efficient. This efficiency comes from the fact that the above model can be treated as a special case of an M/PH/1 queue. Simple explicit results for this M/PH/1 queue can be found in Neuts [10].

In this paper, however, we shall assume that the priority discipline is non-preemptive. So we shall model the computer system by a queueing system with two classes of customers and non-preemptive pri-

ority service discipline. Using an imbedded Markov chain representation of this queueing system and an algorithmic approach discussed by Lucantoni and Neuts [8] and Neuts [9], the steady state joint probability distribution of the number of interactive and batch jobs at a job service completion epoch is derived. Server utilization, mean waiting times, joint probability distribution and mean number of interactive and batch jobs at an arbitrary time epoch are derived using these probabilities, discrete state level crossing analysis [11], and Little's result [6].

2. THE MODEL

The model class considered here is of a single server queueing system to which the arrivals form two independent arrival streams, one from a finite and the other from an infinite population source. The time needed by each of the M members in the finite source, to submit a job (hereafter called interactive job) is exponential with mean $1/\lambda_1$. The jobs from the infinite source (hereafter called batch jobs) arrive to the server according to a Poisson process with rate λ_0. The service requirements of these jobs at the server are general with probability distribution function $B_0(\cdot)$ for batch and $B_1(\cdot)$ for interactive jobs with means $1/\mu_0$ and $1/\mu_1$, respectively. The service discipline is non-preemptive priority. In this paper we consider two cases, one in which the interactive jobs have higher priority over the batch jobs and the other in which these priorities are reversed.

It is routinely verified that the number of interactive and batch jobs just after a service completion, in this model, forms a Markov chain. Let $\{(i,j),\ i\geq 0,\ M\geq j\geq 0\}$ represent the state space with i representing the number of batch and j representing the number of interactive jobs. The transition probability matrix (TPM) P, of this Markov chain is of the form:

$$P = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & B_4 & \cdot & \cdot & \cdot & \cdot \\ A_0 & A_1 & A_2 & A_3 & A_4 & \cdot & \cdot & \cdot & \cdot \\ 0 & A_0 & A_1 & A_2 & A_3 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & A_0 & A_1 & A_2 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & A_0 & A_1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & A_0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where the matrices A_r, $r\geq 0$ and B_r, $r\geq 0$ are all square substochastic matrices of order M+1. The matrices

$$B = \sum_{r=0}^{\infty} B_r \qquad \text{and} \qquad A = \sum_{r=0}^{\infty} A_r$$

are both stochastic.

Let $A_r(i,j)$ and $B_r(i,j)$, $i = 0,1,2,\ldots,M$; $j = 0,1,2,\ldots,M$, be the $(i+1,j+1)$th entry of the A_r and B_r matrices, respectively. Then if $\underset{\sim}{X}(n)$ is the state of the Markov chain at the n-th transition,

$$A_r(i,j) = P\{\underset{\sim}{X}(n+1) = (k+r-1,j)\,|\,\underset{\sim}{X}(n) = (k,i)\},\ k > 0$$

and

$$B_r(i,j) = P\{\underset{\sim}{X}(n+1) = (r,j)\,|\,\underset{\sim}{X}(n) = (0,i)\}$$

for all $i,j = 0,1,2,\ldots,M$ and $n \geq 0$. The exact values of these entries will depend on λ_0, λ_1, $B_0(\cdot)$, $B_1(\cdot)$, and the priority assignment. Before discussing these entries any further, we will first outline the algorithmic approach used to analyze the Markov chain with the structure of P for the TPM. This approach, as a general methodology, was introduced by Lucantoni and Neuts [8] and Neuts [9]. In this section, however, we will present the necessary results only.

Let $\underset{\sim}{x}$ be the steady state probability vector with the partion $\underset{\sim}{x} = (\underset{\sim}{x}_0, \underset{\sim}{x}_1, \ldots)$ where $\underset{\sim}{x}_i$, $i \geq 0$ are all (M+1)-vectors. These steady state probabilities are the solution to the set of equations,

$$\begin{aligned} \underset{\sim}{x}_0 B_0 + \underset{\sim}{x}_1 A_0 &= \underset{\sim}{x}_0 \\ \underset{\sim}{x}_0 B_k + \sum_{r=1}^{k+1} \underset{\sim}{x}_r A_{k+1-r} &= \underset{\sim}{x}_k, \quad k \geq 1 \end{aligned} \tag{1}$$

with the normalizing condition

$$\underset{\sim}{x}\,\underset{\sim}{e} = \sum_{i=0}^{\infty} \underset{\sim}{x}_i \underset{\sim}{e} = 1 \ ,$$

where $\underset{\sim}{e} = (1,1,\ldots)'$ is of appropriate dimension. Next we present the condition for the existence of a steady state solution. Assuming that the matrix A is irreducible, let $\underset{\sim}{\pi}$ be the solution to

$$\underset{\sim}{\pi} A = \underset{\sim}{\pi} \ , \ \underset{\sim}{\pi}\,\underset{\sim}{e} = 1 \tag{2}$$

and $\underset{\sim}{\beta}$ be given by

$$\underset{\sim}{\beta} = \sum_{r=1}^{\infty} r\, A_r \underset{\sim}{e} \ . \tag{3}$$

Then for stationarity (see Neuts [9]),

$$\rho = \underset{\sim}{\pi}\,\underset{\sim}{\beta} < 1 \ . \tag{4}$$

If this condition is satisfied, the steady state joint probability distribution for this Markov chain is given by (see [8]),

$$\left.\begin{aligned} \underset{\sim}{x}_0 &= \frac{\underset{\sim}{d}}{\underset{\sim}{d} \cdot \underset{\sim}{d}^*} \\ \underset{\sim}{x}_1 &= \frac{\underset{\sim}{k}}{\underset{\sim}{k} \cdot \underset{\sim}{k}^*} \end{aligned}\right\} \tag{5}$$

where these quantities $\underset{\sim}{d}$, $\underset{\sim}{k}$, $\underset{\sim}{d}^*$, and $\underset{\sim}{k}^*$ are defined in the appendix. The other elements of the steady state probability vector are obtained using the iterative equations

$$\left.\begin{aligned} \underset{\sim}{x}_k(0) &= \underset{\sim}{b}'_k \\ \underset{\sim}{x}_k(n+1) &= \underset{\sim}{b}'_k + \sum_{r=2}^{k-1} \underset{\sim}{x}_r(n+1)A'_{k+1-r} + \underset{\sim}{x}_{k+1}(n)A'_0 \end{aligned}\right\} \quad (6)$$

where

$$\underset{\sim}{b}'_k = (\underset{\sim}{x}_0 B_k + \underset{\sim}{x}_1 A_k)(I-A_1)^{-1}$$

and

$$A'_r = A_r(I-A_1)^{-1}, \quad r \geq 0 \quad .$$

These iterations of $\underset{\sim}{x}_k(n)$ are continued until a sufficient accuracy is achieved for $\underset{\sim}{x}_k$. The marginal probabilities are

$$x_i = \underset{\sim}{x}_i \underset{\sim}{e}, \quad i \geq 0$$

and (7)

$$\underset{\sim}{x} = \sum_{i=0}^{\infty} \underset{\sim}{x}_i$$

for batch and interactive jobs, respectively.

It can be shown that (see equation (40) of [8])

$$\underset{\sim}{x} = \underset{\sim}{x}(1) = [\underset{\sim}{x}_0 \sum_{r=1}^{\infty} B_r - \underset{\sim}{x}_1 A_0](I-A+\Pi)^{-1} + (1-\underset{\sim}{x}_0 \underset{\sim}{e})\underset{\sim}{\pi} \quad (8)$$

where $\underset{\sim}{x}(Z) = \sum_{r=1}^{\infty} \underset{\sim}{x}_r Z^r$ and Π is an $(M+1)x(M+1)$ matrix with identical columns equal to $\underset{\sim}{\pi}$.

The first moment of the stationary distribution is

$$\underset{\sim}{X}^{(1)}(1) = \{-X(1)[I-\sum_{r=1}^{\infty} rA_r] + \underset{\sim}{x}_0 \sum_{r=1}^{\infty} B_r$$

$$+ \underset{\sim}{x}_0 \sum_{r=1}^{\infty} rB_r - \underset{\sim}{x}_1 A_0\}(I-A+\Pi)^{-1} + (\underset{\sim}{X}^{(1)}(1)\underset{\sim}{e})\underset{\sim}{\pi}, \tag{9}$$

where $\underset{\sim}{X}(1)$ and Π are as defined in (8) and $\underset{\sim}{X}^{(1)}(1)\underset{\sim}{e}$ is defined in the appendix.

Next, using these results and the discrete state level crossing analysis [11], we will derive the steady state joint and marginal probability distributions at an arbitrary time epoch.

Let $\underset{\sim}{q}$ be the steady state joint probability vector at an arbitrary time epoch. Here $\underset{\sim}{q} = (\underset{\sim}{q}_0, \underset{\sim}{q}_1, \ldots)$ and $\underset{\sim}{q}_i = (q_{i0}, q_{i1}, \ldots, q_{iM})$. Then, the rate of upcrossings from the compound state $\{(i,j),\ i\leq m,\ j\leq n\}$ is (see [11]),

$$\lambda_0 \sum_{j=0}^{n} q_{mj} + n\lambda_1 \sum_{i=0}^{m} q_{in}, \tag{10}$$

since the arrivals are Markovian. Now let x^r_{ij} be the steady state probability that a type r (r=0 for batch and r=1 for interactive job) departing job sees i batch and j interactive jobs in the system. Then the rate of downcrossings into the compound state $\{(i,j),\ i\leq m,\ j\leq n\}$ is (see [11]),

$$\rho_0\mu_0 \sum_{j=0}^{n} x^0_{mj} + \rho_1\mu_1 \sum_{i=0}^{m} x^1_{in}\ , \tag{11}$$

where ρ_i is the fraction of time the server is busy with type i jobs. Now equating the rates of up- and downcrossings, from (10) and (11)

we get

$$\lambda_0 \sum_{j=0}^{n} q_{mj} + n\lambda_1 \sum_{i=0}^{m} q_{in} = \rho_0\mu_0 \sum_{j=0}^{n} x^0_{mj} + \rho_1\mu_1 \sum_{i=1}^{m} x^1_{in} .$$

Rearranging terms, we get a recursive formula for the unknowns q_{mn}. It is

$$q_{mn} = \frac{1}{\lambda_0+n\lambda_1}\{\rho_0\mu_0 x^0_{mn} + \rho_1\mu_1 x^1_{mn} + \sum_{j=0}^{n-1} (\rho_0\mu_0 x^0_{mj} - \lambda_0 q_{mj}) + \sum_{i=0}^{m-1} (\rho_1\mu_1 x^1_{in} - n\lambda_1 q_{in}) , \quad 0\leq n\leq M,\ m\geq 0 \tag{12}$$

Next we derive the marginal pr...ilities associated at an arbitrary time epoch.

(1) Interactive Jobs

Equating the rates of up- and downcrossings (see page 21 of [11]), we get

$$n\lambda_1 q^1_n = \rho_1\mu_1 x^1_n , \quad 0\leq n<M \tag{13}$$

where $x^1_n = \sum_{i=0}^{\infty} x^1_{in}$ and q^1_n is the steady state probability that there are n interactive jobs in the system at an arbitrary time epoch. Then

$$\left.\begin{aligned} q^1_n &= \frac{\rho_1\mu_1}{n\lambda_1}\, x^1_n , \quad 0\leq n<M \\ \text{and} \qquad q^1_M &= 1 - \sum_{i=0}^{M-1} q^1_i \end{aligned}\right\} \tag{14}$$

(ii) Batch Jobs

Using a similar derivation as above (see [11]), we get

$$\lambda_0 q^0_m = \rho_0 \mu_0 x^0_m , \tag{15}$$

where $x^0_m = \sum_{j=0}^{M} x^0_{mj}$ and q^0_m is the steady state probability that there are m batch jobs in the system at an arbitrary time epoch. Since all batch jobs are serviced, $\rho_0\mu_0 = \lambda_0$. Substituting this in (15), we have

$$q^0_m = x^0_m , \quad m \geq 0 \tag{16}$$

The following section provides equations to evaluate the mean number of jobs in the system at an arbitrary time epoch, and the mean waiting times.

(i) Interactive Jobs

Let the effective interactive job arrival rate be λ^1_e. Then $\lambda^1_e = \rho_1\mu_1$. If L_1 is the mean number of interactive jobs at the server at an arbitrary time epoch, and W_1 is its mean waiting time,

$$\lambda^1_e = (M-L_1)\lambda_1$$

or

$$L_1 = M - \rho_1\mu_1/\lambda_1 \tag{17}$$

Now from (17) and Little's result [6],

$$W_1 = (M/\rho_1\mu_1) - 1/\lambda_1 \,. \tag{18}$$

(ii) Batch Jobs

Let L_0 and N^0 be the mean number of batch jobs in the system at an arbitrary time epoch and just after a batch jobs service completion, and W_0 its mean waiting time. Then

$$L_0 = N^0 \,, \tag{19}$$

since $q^0_m = x^0_m$, and from Little's result,

$$W_0 = L_0/\lambda_0 \tag{20}$$

To use (12), (14), (16)-(19), and (20), we need x^r_{ij}, ρ_r, $r = 0,1$, and N^0. These can be related to $\underset{\sim}{x}$ and $\underset{\sim}{X}^{(1)}(1)$ depending on the priority rule used. We shall derive these relationships later for each case, separately. The following definitions will be used in both cases.

$P_i(n,r:k)$ = P{n batch jobs and r interactive jobs arrive during a time interval which has a probability distribution $B_i(\cdot)$, given there are k interactive jobs left in the source at the beginning of this time interval}.

CASE 1: Interactive Jobs Have Higher Priority Over Batch Jobs

Here we assume that the interactive jobs have higher priority over batch jobs. That is, if there is at least one interactive job available at a service completion epoch, it will go in for service. Then the entries of the matrix A_r are given by,

for $r \geq 1$

$$A_r(i,j) = \begin{cases} P_1(r-1, j+1-i : M-i) & M \geq i > 0;\ M \geq j \geq i-1 \\ 0 & M \geq i > 0;\ i-1 > j \geq 0 \\ P_0(r, j : M) & i = 0;\ M \geq j \geq 0 \end{cases}$$

and for $r=0$

$$A_0(i,j) = \begin{cases} 0 & M \geq i > 0;\ M \geq j \geq 0 \\ P_0(0, j : M) & i = 0;\ M \geq j \geq 0 \end{cases}$$

Similarly, for $r \geq 0$

$$B_r(i,j) = \begin{cases} A_{r+1}(i,j) & M\geq i>0;\ M\geq j\geq 0 \\ a_0P_0(r,j:M)+a_1P_1(r,j:M-1) & ,\ i=0:M-1\geq j\geq 0 \\ a_0P_0(r,M:M) & i=0;\ j=M\ , \end{cases}$$

$$\text{where } a_0 = \frac{\lambda_0}{\lambda_0+M\lambda_1} \text{ and } a_1 = \frac{M\lambda_1}{\lambda_0+M\lambda_1}\ . \tag{21}$$

These probabilities can be evaluated in a recursive manner and the details are given in the appendix. Now to use equations (12), (14), (16)-(19), and (20), we need $\underset{\sim}{x}^0$, $\underset{\sim}{x}^1$, ρ_0, ρ_1, and N^0. We will derive these next.

Since for an interactive job to leave behind i batch and n interactive jobs, an interactive job must go in for service, thus the use of conditional probability and convolution arguments lead to

$$x^1_{in} = \{\sum_{k=1}^{i}\sum_{j=1}^{n+1} x_{kj}A_{i+1-k}(j,n) + \sum_{j=1}^{n+1} x_{0j}B_i(j,n) + a_1x_{00}B_i(0,n)\}/p_1 \tag{22}$$

and

$$x^1_n = \sum_{i=0}^{\infty} x^1_{in}\ , \tag{23}$$

where p_1 is the probability that a departing job is interactive. Now for batch jobs,

$$x^0_{mj} = (\sum_{i=1}^{m+1} x_{i0} A_{m+1-i}(0,j) + a_0 x_{00} B_m(0,j))/p_0 \ , \qquad (24)$$

where p^C is the probability that a departing job is a batch job.

Then

$$x^0_m = \sum_{j=0}^{M} x^0_{mj}$$

$$= \frac{1}{p_0} \{ \sum_{i=1}^{m+1} x_{i0} \sum_{j=0}^{M} A_{m+1-i}(0,j) + a_0 x_{00} \sum_{j=0}^{M} B_m(0,j) \} \ .$$

After some simplification, and using the normalizing condition, we get

$$x^0_m = \{ \sum_{i=1}^{m+1} x_{i0} P(m+1-i) + a_0 x_{00} P(m) \}/p_0 \ , \qquad (25a)$$

where $P(n)$ is the probability that n batch jobs arrive during the service of a batch job, and

$$p_0 = \sum_{i=1}^{\infty} x_{i0} + a_0 x_{00} \ . \qquad (25b)$$

Using the theory of Markov Renewal Processes (see J.J. Hunter [3]), it can be shown that

$$\rho_0 = (\sum_{i=1}^{\infty} x_{i0} + a_0 x_{00}) \frac{1}{\mu_0} / EC \ ,$$

where

$$EC = (\sum_{i=1}^{\infty} x_{i0} + a_0 x_{00}) \frac{1}{\mu_0} + (\sum_{i=0}^{\infty} \sum_{j=1}^{M} x_{ij} + a_1 x_{00}) \frac{1}{\mu_1} + \frac{x_{00}}{\lambda_0 + M\lambda_1}$$

and

$$\rho_1 = \left(\sum_{i=0}^{\infty} \sum_{j=1}^{M} x_{ij} + a_1 x_{00} \right) \frac{1}{\mu_1} / EC,$$

since ECxEN is the mean busy cycle time and $(\sum_{i=1}^{\infty} x_{i0} + a_0 x_{00}) xEN/\mu_0$ and $(\sum_{i=0}^{\infty} \sum_{j=1}^{M} x_{ij} + a_1 x_{00}) xEN/\mu_1$ are the mean time in a busy cycle the server is busy with batch and interactive jobs, respectively, where EN is the mean number of jobs served during a busy cycle. Note that

$$\sum_{i=1}^{\infty} x_{i0} = y_0 - x_{00}$$

and

$$\sum_{i=0}^{\infty} \sum_{j=1}^{M} x_{ij} = 1 - y_0$$

and so from (25b)

$$p_0 = y_0 - x_{00} + a_0 x_{00} \quad . \tag{25c}$$

Next we evaluate N^0. Since

$$N^0 = \sum_{m=0}^{\infty} m \, x^0_m$$

from (25a) we get

$$N^0 = \sum_{m=0}^{\infty} m\{\sum_{i=1}^{m+1} x_{i0} P(m+1-i) + x_{00} a_0 P(m)\}/p_0$$

$$= \{\sum_{i=1}^{\infty} \sum_{m=i-1}^{\infty} m\, x_{i0} P(m-(i-1)) + x_{00} a_0 \sum_{m=0}^{\infty} m\, P(m)\}/p_0$$

$$= \{\sum_{i=1}^{\infty} ((i-1) + \rho_0) x_{i0} + x_{00} a_0 \rho_0\}/p_0$$

$$= \{\sum_{i=1}^{\infty} i\, x_{i0} - (1-\rho_0)(\sum_{i=1}^{\infty} x_{i0}) + a_0 x_{00} \rho_0\}/p_0$$

$$= \{\underset{\sim}{x}^{(1)}(1)(0) - (1-\rho_0)(y_0 - x_{00}) + a_0 x_{00} \rho_0\}/p_0 \;.$$

So

$$L_0 = \{\underset{\sim}{x}^{(1)}(1)(0) - (1-\rho_0)(y_0 - x_{00}) + a_0 x_{00} \rho_0\}/p_0 \tag{26}$$

and hence $W_0 = L_0/\lambda_0$ can be obtained once we evaluate $\underset{\sim}{x}$. We can use (5) and (6) to obtain this provided A is irreducible and inequality (4) holds. From the definition of A_r, it is easily verified that the (i+1,j+1) the entry A(i,j) of A is given by

$$A(i,j) = \begin{cases} {}_0\int^{\infty} \binom{M-i}{j+1-i} (1-e^{-\lambda_1 t})^{j+1-i} e^{-(M-j-1)\lambda_1 t} dB_1(t), & m \geq i > 0,\ m \geq j \geq i-1 \\ {}_0\int^{\infty} \binom{M}{j} (1-e^{-\lambda_1 t})^{j} e^{-(M-j)\lambda_1 t} dB_0(t), & i=0;\ M \geq j \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Similarly, β is given by

$$\beta = (\lambda_0/\mu_0, \lambda_0/\mu_1, \lambda_0/\mu_1, \ldots, \lambda_0/\mu_1)$$

Then the necessary and sufficient condition for the existance of x is

$$\rho = \pi\beta < 1 ,$$

where

$$\pi A = \pi .$$

CASE 2: Batch Jobs Have Higher Priority Over Interactive Jobs

In this section we assume that the batch jobs have higher priority over interactive jobs. That is, if there is at least one batch job available at a service completion epoch, it will go in for service. From the definition of $P_i(n,r:k)$ it is easily verified that,

<u>for $r \geq 0$</u>

$$A_r(i,j) = \begin{cases} P_0(r, j-i : M-i) , & M \geq i \geq 0;\ M \geq j \geq i \\ \\ 0 & M \geq i \geq 0;\ i > j \geq 0 \end{cases}$$

and

$$
B_r(i,j) = \begin{cases} P_1(r, j+1-i : M-i) & M \geq i > 0;\ M \geq j \geq i-1 \\ a_0 P_0(r, j:M) + a_1 P_1(r, j:M-1)\ , & i=0; M-1 \geq j \geq 0 \\ a_0 P_0(r, M:M) & i=0;\ j=M \\ 0 & \text{otherwise} \end{cases}
$$

As indicated in the previous case, these probabilities can be evaluated recursively. Now we will derive expressions for $\rho_0, \rho_1, \xi^0, \xi^1$, and N^0.

Similar to the previous case, using Markov Renewal Theory, we get

$$
\rho_0 = \Big(\sum_{m=1}^{\infty} (x_m) + a_0 x_{00} \Big) \frac{1}{\mu_0} / EC\ ,
$$

where

$$
EC = \Big(\sum_{j=1}^{M} x_{0j} + a_1 x_{00} \Big) \frac{1}{\mu_1} + \Big(\sum_{m=1}^{\infty} (x_m) + a_0 x_{00} \Big) \frac{1}{\mu_0} + \frac{x_{00}}{\lambda_0 + M\lambda_1}
$$

and

$$
\rho_1 = \Big(\sum_{j=1}^{M} x_{0j} + a_1 x_{00} \Big) \frac{1}{\mu_1} / EC\ .
$$

Note that $\sum_{m=1}^{\infty} (x_m) = 1 - x_0$ and $\sum_{j=1}^{M} x_{0j} = x_0 - x_{00}$.

The relationships for $\underset{\sim}{x}^0$ and $\underset{\sim}{x}^1$ with respect to $\underset{\sim}{x}$ is presented next. These results follow from simple convolution arguments and conditional probabilities.

$$x^0_{mj} = \{ \sum_{i=1}^{m+1} \sum_{k=0}^{j} x_{ik} P_0(m+1-i,\ j-k:M-k)$$

$$+ x_{00} a_0 P_0(m,j:M) \}/p_0 \ ,$$

where p_0 is the probability that a departing job is . batch job, and so

$$x^0_m = \sum_{j=0}^{M} x^0_{mj}$$

$$= \{ \sum_{i=1}^{m+1} x_i P(m+1-i) + x_{00} a_0 P(m) \}/p_0 \ , \qquad (27)$$

where $p_0 = 1 - x_0 + a_0 x_{00}$. Therefore

$$N^0 = \sum_{m=0}^{\infty} m\, x^0_m$$

$$= \{ \sum_{m=0}^{\infty} m \sum_{i=1}^{m+1} x_i P(m+1-i) + \sum_{m=0}^{\infty} a_0 x_{00} m P(m) \}/p_0$$

$$= \{ \sum_{i=1}^{\infty} \sum_{m=i-1}^{\infty} m x_i P(m-(i-1)) + a_0 x_{00} \rho_0 \}/p_0$$

$$= \{ \sum_{i=1}^{\infty} (i-1+\rho_0) x_i + a_0 x_{00} \rho_0 \}/p_0 \ .$$

So

$$L_0 = \{\underset{\sim}{\chi}^{(1)}(1)\underset{\sim}{e} - (1-\rho_0)(1-x_0) + a_0 x_{00}\rho_0\}/p_0 \tag{28}$$

and $W_0 = L_0/\lambda_0$ can be obtained once $\underset{\sim}{x}$ is known.

Now for x^1_{in} we have from convolution arguments,

$$x^1_{in} = \{\sum_{j=1}^{n+1} x_{0j} P_1(i,n+1-j:M-j) + a_1 x_{00} P_1(i,n:M-1)\}/p_1 \ , \tag{29}$$

where p_1 is the probability that a departing job is interactive and

$$x^1_n = \sum_{i=0}^{\infty} x^1_{in} \ , \ M \geq n \geq 0 \ . \tag{30}$$

Now we will specify the condition for the existence of $\underset{\sim}{x}$. Considering A, it is clear that A is an upper triangular matrix whose entries, with notation as defined in the previous case, are

$$A(i,j) = \begin{cases} \int_0^\infty \binom{M-i}{j-i}(1-e^{-\lambda_1 t})^{j-i} e^{-\lambda_1 (M-j)t} dB_0(t), & M \geq i \geq 0;\ M \geq j \geq i \\ \\ 0 & \text{otherwise} \end{cases}$$

Clearly A is reducible and therefore the condition (4) cannot be directly used. However, using an analysis similar to that of Lucantoni [7] page 7, it can be established that the necessary and sufficient condition for stability is

$$\lambda_0/\mu_0 < 1 \ .$$

3. NUMERICAL RESULTS

The algorithm generating these performance measures was implemented in APL. Some sample results for different input parameters λ_0, λ_1, μ_0, μ_1, and M for exponential service times are shown in Tables 1 and 2.

Batch Jobs Arrival Rate λ_0	Mean number of jobs in the system at an arbitrary time epoch when higher priority is given to Interactive Jobs		Batch Jobs	
	Batch	Interactive	Batch	Interactive
.05	0.2835	0.5875	0.2567	0.6841
.075	0.5133	0.7376	0.4394	0.9565
.10	0.8569	0.8878	0.6821	1.2890
.125	1.4190	1.0380	1.0210	1.6920
.15	2.4879	1.1881	1.5265	2.1713
.175	5.2676	1.3383	2.3663	2.7365

Batch jobs service rate $\mu_0 = 0.25$
Number of Interactive terminals N = 5
Interactive job arrival rate $\lambda_1 = 0.1$
Interactive jobs service rate $\mu_1 = 2.0$

TABLE 1

Number of Interactive Terminals	Mean number of jobs in the system at an arbitrary time epoch when higher priority is given to			
	Interactive Jobs		Batch Jobs	
	Batch	Interactive	Batch	Interactive
5	0.2835	0.5875	0.2567	0.6841
6	0.2939	0.7327	0.2580	0.8552
7	0.3038	0.8895	0.2591	1.0407
8	0.3184	1.0598	0.2604	1.2422

Batch jobs arrival rate $\lambda_0 = 0.05$

Batch jobs service rate $\mu_0 = 0.25$

Interactive jobs arrival rate $\lambda_1 = 0.1$

Interactive jobs service rate $\mu_1 = 2.0$

TABLE 2

4. CONCLUSION

In this paper we have modelled the computer system with batch and interactive jobs as a single server queueing system with two classes of customers and non-preemptive priority service discipline. The system performance measures were numerically evaluated using an algorithmic approach proposed by Neuts [9] and a discrete state level crossing analysis (see [11]).

APPENDIX

(a) Recursive Evaluation of Transition Probabilities

Define

$P_i(n,r:k) = P\{$n batch jobs and r interactive jobs arrive during a time interval which has probability distribution $B_i(\cdot)$ given there are k interactive jobs left in the source at the beginning of this time interval$\}$.

That is

$$P_i(n,r:k) = {}_0\int^\infty \frac{e^{-\lambda_0 t}(\lambda_0 t)^n}{n!} \binom{k}{r}(1-e^{-\lambda_1 t})^r e^{-(k-r)\lambda_1 t} dB_i(t), \quad i=0,1$$

$$= \frac{\lambda_0^n}{n!}\binom{k}{r} {}_0\int^\infty e^{-\lambda_0 t} t^n \left(\sum_{j=0}^{r} \binom{r}{j}(-1)^j e^{-j\lambda_1 t}\right) e^{-(k-r)\lambda_1 t} dB_i(t) \qquad \text{(A1)}$$

$$= \frac{\lambda_0^n}{n!}\binom{k}{r} \sum_{j=0}^{r} (-1)^j \binom{r}{j} {}_0\int^\infty t^n e^{-(\lambda_0+(k+j-r)\lambda_1 t} dB_i(t)$$

$$= (-1)^n \frac{\lambda_0^n}{n!}\binom{k}{r} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \tilde{B}_i^{(n)}(\lambda_0+(k+j-r)\lambda_1) ,$$

where

$$\tilde{B}_i^{(n)}(a) = \lim_{s\to a} \frac{d^n}{ds^n} \tilde{B}_i(s)$$

and

$$\tilde{B}_i(s) = \int_0^\infty e^{-sx} dB_i(x) \quad , \quad i=0,1 \ .$$

Now consider

$$P_i(n,r:k+1) = (-1)^n \frac{\lambda_0^n}{n!} {k+1 \choose r} \sum_{j=0}^{r} (-1)^j {r \choose j} \tilde{B}_i^{(n)}(\lambda_0+(k+1+j-r)\lambda_1) \ .$$

Using the substitution

$${r \choose j} = {r-1 \choose j} + {r-1 \choose j-1}$$

we get, after some manipulation,

$$P_i(n,r:k+1) = (-1)^n \frac{\lambda_0^n}{n!} {k+1 \choose r} \sum_{j=0}^{r-1} (-1)^j {r-1 \choose j} \tilde{B}_i^{(n)}(\lambda_0+(k+j-(r-1)\lambda_1)$$

$$- (-1)^n \frac{\lambda_0^n}{n!} {k+1 \choose r} \sum_{j=0}^{r-1} (-1)^j {r-1 \choose j} \tilde{B}_i^{(n)}(\lambda_0+(k+1+j-(r-1)\lambda_1)$$

$$= \{ {k+1 \choose r} / {k \choose r-1} \} P_i(n,r-1:k) - \{ {k+1 \choose r} / {k+1 \choose r-1} \} P_i(n,r-1:k+1)$$

That is,

$$P_i(n,r:k+1) = (\frac{k+1}{r}) P_i(n,r-1:k) - (\frac{k+2-r}{r}) P_i(n,r-1:k+1) \ , \tag{A2}$$

$$n \geq 0; \quad r \leq k \leq M$$

Note that

$$P_i(n,r:k+1) = 0 \quad \text{for all } r>k+1$$

and

$$P_i(n,0:k) = (-1)^n \frac{\lambda_0^n}{n!} \tilde{B}_i^{(n)}(\lambda_0+k\lambda_1), \; M\geq k\geq 0 \; .$$

Then using (A2), $P_i(n,r:k)$ can be calculated in a recursive fashion. That is, for $r=1$, calculate $P_i(n,1:k)$, $k = 1,2,\ldots,M$, using (A2) and then $r=2$, etc.

(b) Quantities Needed in the Analysis of the Imbedded Markov Chain (taken from Lucantoni and Neuts [8]).

$$A^*(Z) = \sum_{r=0}^{\infty} Z^r A_r$$

$$\underset{\sim}{\chi}(Z) = \sum_{r=1}^{\infty} Z^r \underset{\sim}{\chi}_r$$

$$G(Z) = \sum_{r=0}^{\infty} ZA_r G^r(Z) = ZA^*(G(Z))$$

$$G = \sum_{r=0}^{\infty} A_r G^r$$

$$\underset{\sim}{\pi} A = \underset{\sim}{\pi} \qquad \underset{\sim}{\pi}\underset{\sim}{e} = 1$$

$$\underset{\sim}{\beta} = \sum_{r=1}^{\infty} rA_r\underset{\sim}{e}$$

$$\underset{\sim}{g}G = \underset{\sim}{g} \quad , \quad \underset{\sim}{g}\underset{\sim}{e} = 1$$

$\underset{\sim}{G}$ is a square matrix with all row vectors identical to $\underset{\sim}{g}$.

$\Delta(\underset{\sim}{\beta})$ is a diagonal matrix of order M+1 with diagonal entries $\beta_0, \beta_1, \ldots, \beta_M$.

$$\underset{\sim}{\mu} = (I-G+\tilde{G})[I-A+\tilde{G}-\Delta(\underset{\sim}{\beta})\tilde{G}]^{-1}\underset{\sim}{e}$$

$$H(Z) = Z[I-\sum_{r=1}^{\infty} ZA_r G^{r-1}(A)]^{-1}A_0$$

$$L(Z) = ZB_0 + \sum_{r=1}^{\infty} ZB_r G^{r-1}(Z)H(Z)$$

$$K(Z) = ZA_0(I-ZB_0)^{-1}\sum_{r=1}^{\infty} ZB_r G^{r-1}(Z) + \sum_{r=1}^{\infty} ZA_r G^{r-1}(Z)$$

$$\underset{\sim}{d}\,L(1) = \underset{\sim}{d} \quad \text{and} \quad \underset{\sim}{d}\underset{\sim}{e} = 1$$

$$\underset{\sim}{k}K(1) = \underset{\sim}{k} \quad \text{and} \quad \underset{\sim}{k}\underset{\sim}{e} = 1$$

$$\underset{\sim}{d}^{*} = L^{1}(1)\underset{\sim}{e}$$

$$\underset{\sim}{h}^{*} = H^{1}(1)\underset{\sim}{e}$$

$$k^{*} = k^{1}(1)\underset{\sim}{e}$$

$$\underset{\sim}{h}^* = (I-\sum_{r=1}^{\infty} A_r G^{r-1})^{-1}\{\underset{\sim}{e} + [\sum_{r=1}^{\infty} A_r - \sum_{r=1}^{\infty} A_r G^{r-1} + \sum_{r=2}^{\infty} (r-1) A_r \tilde{G}](I-G+\tilde{G})^{-1}\underset{\sim}{\mu}\},$$

$$\underset{\sim}{d}^* = \underset{\sim}{e} + \sum_{r=1}^{\infty} B_r G^{r-1}\underset{\sim}{h}^* + [\sum_{r=1}^{\infty} B_r - \sum_{r=1}^{\infty} B_r G^{r-1} + \sum_{r=2}^{\infty} (r-1) B_r \tilde{G}](I-G+\tilde{G})^{-1}\underset{\sim}{\mu}$$

$$\underset{\sim}{k}^* = \underset{\sim}{e} + A_0(I-B_0)^{-1}\underset{\sim}{e} + \{A_0(I-B_0)^{-1}[\sum_{r=1}^{\infty} B_r - \sum_{r=1}^{\infty} B_r G^{r-1}$$

$$+ \sum_{r=2}^{\infty} (r-1) B_r \tilde{G}] + \sum_{r=1}^{\infty} A_r - \sum_{r=1}^{\infty} A_r G^{r-1} + \sum_{r=2}^{\infty} (r-1) A_r \tilde{G}\ (I-G+\tilde{G})^{-1}\underset{\sim}{\mu}$$

$$A^*_n(Z) = \frac{d^n}{dZ^n} A^*(Z)$$

$$\delta^{(0)}(1) = 1,\ \underset{\sim}{\mu}^{(0)}(1) = \underset{\sim}{e},\ \underset{\sim}{\chi}^{(0)}(1) = \underset{\sim}{\pi},\ \delta^{(1)} = \underset{\sim}{\pi} A_1^*(1)\underset{\sim}{e}$$

$$\underset{\sim}{\mu}^{(1)}(1) = (I-A+\Pi)^{-1}\underset{\sim}{\beta} - \delta^{(1)}\underset{\sim}{e}$$

$$\underset{\sim}{\chi}^{(1)}(1) = \underset{\sim}{\pi} A_1^*(1)(I-A+\Pi)^{-1} - \delta^{(1)}\underset{\sim}{\pi}$$

for $n \geq 2$

$$\delta^{(n)}(1) = \sum_{r=1}^{n} \binom{n}{r} \underset{\sim}{\pi} A_r^*(1) \underset{\sim}{\mu}^{(n-r)}(1) - \sum_{r=1}^{n-1} \underset{\sim}{\pi} \underset{\sim}{\mu}^{(n-r)}(1) \delta^{(r)}(1)$$

$$\underset{\sim}{\mu}^{(n)}(1) = (I-A+\Pi)^{-1} \sum_{r=1}^{n} \binom{n}{r} [A_r^*(1) - \delta^{(r)}(1) I] \underset{\sim}{\mu}^{(n-r)}(1)$$

$$- \Big[\sum_{r=1}^{n-1} \binom{n}{r} \underset{\sim}{\chi}^{(r)}(1) \underset{\sim}{\mu}^{(n-r)}(1)\Big] \underset{\sim}{e}$$

$$\underset{\sim}{\chi}^{n}(1) = \sum_{r=0}^{n-1} \binom{n}{r} \underset{\sim}{\chi}^{(r)}(1) [A_{n-r}^*(1) - \delta^{(n-r)}(1) I] (I-A+\Pi)^{-1}$$

$$\underset{\sim}{\chi}^{(1)}(1)\underset{\sim}{e} = \frac{1}{2(1-\rho)} \Big\{ 2\underset{\sim}{x}_0 \sum_{k=1}^{\infty} kB_k \underset{\sim}{e} + 2\underset{\sim}{x}_0 \sum_{k=1}^{\infty} B_k \underset{\sim}{\mu}^{(1)}(1) + \underset{\sim}{x}_0 \sum_{k=2}^{\infty} k(k-1) B_k \underset{\sim}{e}$$

$$+ 2\underset{\sim}{x}_0 \sum_{k=1}^{\infty} kB_k \underset{\sim}{\mu}^{(1)}(1) + \underset{\sim}{x}_0 \sum_{k=1}^{\infty} B_k \underset{\sim}{\mu}^{(2)}(1) - 2\underset{\sim}{x}_1 A_0 \underset{\sim}{\mu}^{(1)}(1)$$

$$- \underset{\sim}{x}_1 A_0 \underset{\sim}{\mu}^{(2)}(1) + \underset{\sim}{\chi}(1) \underset{\sim}{e} \delta^{(2)}(1) \Big\} - \underset{\sim}{\chi}(1) \underset{\sim}{\mu}^{(1)}(1).$$

The evaluation G is carried out using the recursion (see page 6 of [8]).

$$\hat{G}(0) = (I-A_1)^{-1} A_0,$$

$$\hat{G}(k+1) = \sum_{\substack{v=0 \\ v \neq 1}}^{\infty} (I-A_1)^{-1} A_r G^r(k) \ , \text{ for } k \geq 0$$

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